International Journal of Theoretical Physics, Vol. 32, No. 1, 1993

Autonomous Systems, Dynamical Systems, LPTI Symmetries, Topology of Trajectories, and Periodic Solutions

G. Gaeta[1]

Received February 3, 1992

In the case of autonomous dynamical systems, it is better to base symmetry considerations on trajectories than on full solutions. In this setting topological arguments can be used; a special role is played in this context by time-independent Lie-point symmetries. As an application of this approach, we obtain results on the existence of stationary and/or periodic solutions.

1. INTRODUCTION

This paper originated from the desire to extend the results relating linear symmetries of ordinary differential equations to their periodic solutions (Cicogna and Gaeta, 1989) to the case of general geometric, i.e., possibly nonlinear, or, as they are called, Lie-point (LP) symmetries (Olver, 1986; Bluman and Kumei, 1989; Ovsjannikov, 1962; Stephani, 1989; Gaeta, 1992); the reader is assumed to be familiar with these and with the symmetry approach to differential equations.

It turns out that in the case of autonomous dynamical systems, it is more profitable to consider only time-independent Lie-point symmetries; the advantage is equivalent, and indeed strongly related, to the one given by considering the reduced phase space in which the t coordinate is projected out, rather than the full one.

We begin therefore by discussing LP time-independent (LPTI) symmetries of first-order autonomous ODEs and of their solutions; consistent with our general approach, we focus on trajectories rather than on full solutions, which leads to definitions (slightly) different from the usual ones.

[1]Centre de Physique Theorique, Ecole Polytechnique, F-91128 Palaiseau, France.

0020-7748/93/0100-0191$07.00/0

We also point out that in this case (first-order ODE) the set of vector fields leaving the equation invariant has not only the usual structure of a Lie algebra, but also that of a module (over the algebra of constants of motion), a fact which is often overlooked, leading to confusion about the dimension and set of generators of the set of vector fields (these are different if we consider the set as an algebra or as a module).

We discuss then the relation between Lie-point time-independent symmetries and the topology of solutions, and more precisely of trajectories of solutions in the reduced phase space; this also shows the advantages of considering LPTI symmetries instead of general LP ones, and trajectories instead of full solutions (see Lemmas I and II): in short, LPTI symmetries preserve the topology of trajectories.

We then discuss periodic and stationary solutions, through a reduction argument (see Lemma III); in particular, when the manifolds to which we reduce are compact, which could be ensured by an assumption on F we make, we can extend the results of Cicogna and Gaeta (1989); see Lemma III′ and its corollaries.

We stress that our method and results do not require the knowledge of the whole symmetry of the differential equation in question (which could be difficult to obtain in full completeness), but can instead be fruitfully applied even if one knows only one symmetry of the equation.

2. SYMMETRY OF EQUATIONS

We will consider a smooth dynamical system on the smooth manifold M embedded in $\mathbf{R}^N$, i.e.,

$$\dot{x}=F(x) \tag{1}$$

with $x\equiv x(t)\in M$, $t\in\mathbf{R}_+$, $F\colon M\to TM$; the vector field is supposed to be (i) smooth, and (ii) such that there is a compact set $K\subset M$ invariant under the flow of (1).

Throughout this note, smooth will mean $\mathscr{C}^\infty$; all the functions, manifolds, etc., are assumed to be smooth.

Since (1) is an autonomous equation, we will find it profitable to take care of this specificity in the discussion of its symmetry properties. Indeed, (1) can be identified with its *solution manifold* (Olver, 1986; Bluman and Kumei, 1989; Gaeta, 1992),

$$S=\{(t,x,\dot{x})/\dot{x}=F(x)\}\subset\mathbf{R}_+\times TM \tag{2}$$

and the symmetry algebra $\mathscr{G}$ of (1) is the algebra of vector fields η on $\mathbf{R}_+\times M$ such that their first prolongation (Olver, 1986; Bluman and Kumei, 1989;

Gaeta, 1992) $\eta^{(1)}$ satisfies

$$\eta^{(1)}: \quad S \to TS \tag{3}$$

In the case of (1), S is obviously a direct product

$$S = \mathbf{R}_+ \times S_0; \qquad S_0 = \{(x, \dot{x})/\dot{x} = F(x)\} \subset TM \tag{4}$$

so that it is natural to concentrate on S_0, which we do.

Let us consider the algebra $\mathscr{M}$ of diffeomorphisms of M; the Lie-point time-independent (LPTI) symmetry algebra $\mathscr{G}_0$ of (1) will just be

$$\mathscr{G}_0 = \{\gamma \in \mathscr{M}/\gamma^{(1)}: S_0 \to TS_0\} \tag{5}$$

where $\gamma^{(1)}$ is the prolongation of γ. If γ is given by the vector field

$$\gamma = \phi^i(x) \frac{\partial}{\partial x^i} \equiv \phi^i(x) \partial_i \tag{6}$$

then the general prolongation formula gives

$$\gamma^{(1)} = \phi^i(x) \frac{\partial}{\partial x^i} + \Phi^i(x, \dot{x}) \frac{\partial}{\partial \dot{x}^i} \equiv \phi^i(x) \frac{\partial}{\partial x^i} + \dot{x}^j \partial_j \phi^i(x) \frac{\partial}{\partial \dot{x}^i} \tag{7}$$

Considering the restriction of this to S_0, i.e., to $\dot{x} = F(x)$, we have immediately that $\gamma \in \mathscr{G}_0$ if and only if

$$\{F, \phi\} = 0 \tag{8}$$

where we have defined the bracket $\{\cdot, \cdot\}: \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ as

$$\{\Psi, \chi\}^i = \Psi^j \partial_j \chi^i - \chi^j \partial_j \Psi^i \tag{9}$$

Notice that $\gamma_0 = F^i(x) \partial_i$ satisfies (8) and is therefore, as it should be, in $\mathscr{G}_0$. For completeness, we also notice that if one considers the algebra $\tilde{\mathscr{M}}$ of time-dependent diffeomorphisms of M (then $\mathscr{G}_0$ is replaced by $\tilde{\mathscr{G}}_0$) and retain the forms (6), (9) for $\gamma = \phi(x, t) \partial_x \in \tilde{\mathscr{M}}$ and $\{\cdot, \cdot\}: \tilde{\mathscr{M}} \times \tilde{\mathscr{M}}$, then (8) is replaced by

$$\phi_t + \{F, \phi\} = 0 \tag{8'}$$

It is also worth remarking explicitly that in the case of first-order ODEs the symmetry algebra $\mathscr{G}$ and the LPTI one $\mathscr{G}_0$ have the algebraic structure of a *module* over the algebras I, respectively I_0, of constants of motion (respectively, time-independent constants of motion), as follows immediately from the prolongation formula. In our setting, it is immediate to see that for $\alpha(x)\in I_0$ (which just means $F^i\,\partial_i\alpha=0$), one has $\{F,\alpha\phi\}=\alpha\{F,\phi\}$, so that

$$\gamma_1,\ldots,\gamma_h\in\mathscr{G}_0;\qquad \alpha_1,\ldots,\alpha_h\in I_0\Rightarrow\sum_{i=1}^{h}\alpha_i(x)\gamma_i\in\mathscr{G}_0 \tag{10}$$

and similarly for $\tilde{\mathscr{G}}_0$.

3. SYMMETRY OF SOLUTIONS

Let us now consider a solution $x=f(t)$ of (1), $f\colon \mathbf{R}_+\to M$. This defines a *graph* Γ_f in $\mathbf{R}_+\times M$, i.e.,

$$\Gamma_f=\{(t,x)\in R_+\times M/x=f(t)\}\subset\mathbf{R}_+\times M \tag{11}$$

For an autonomous equation, it is again natural to consider instead the projection of this to M, i.e., the *trajectory* Θ_f of $x=f(t)$,

$$\Theta_f=\{x\in M/\exists t\in\mathbf{R}_+:x=f(t)\}\subset M \tag{12}$$

The curves $\Gamma_f\subset\mathbf{R}_+\times M$ and $\Theta_f\subset M$ are naturally lifted to curves $\Gamma_f^{(1)}\subset\mathbf{R}_+\times TM$ and $\Theta_f^{(1)}\subset TM$,

$$\Gamma_f^{(1)}=\{(t;x,p)\in R_+\times TM/x=f(t),p=f'(t)\}\subset R_+\times TM \tag{13}$$

$$\Theta_f^{(1)}=\{(x,p)\in TM/\exists t\in R_+:x=f(t),p=f'(t)\}\subset TM \tag{14}$$

If κ is the projection from $\mathbf{R}_+\times M$ to M and $d\kappa$ is that from $\mathbf{R}_+\times TM$ to TM, so that $\kappa\cdot(t,x)=x$ and $d\kappa\cdot(t,x,p)=(x,p)$, and π is the projection from TM to M, we have

$$\begin{array}{ccc}\Gamma_f^{(1)} & \xrightarrow{I\times\pi} & \Gamma_f\\ \downarrow{\scriptstyle d\kappa} & & \downarrow{\scriptstyle\kappa}\\ \Theta_f^{(1)} & \xrightarrow{\pi} & \Theta_f\end{array} \tag{15}$$

Given a function $f\colon\mathbf{R}_+\to M$ [in particular, a solution to (1)], we define its LPTI symmetry algebra as

$$\mathscr{G}_f=\{\gamma\in\mathscr{M}/\gamma:\Theta_f\to T\Theta_f\} \tag{16}$$

Notice that $\gamma:\Theta_f\to T\Theta_f$ if and only if $\gamma^{(1)}:\Theta_f^{(1)}\to T\Theta_f^{(1)}$, so the above is equivalent to

$$\mathscr{G}_f=\{\gamma\in\mathscr{M}/\gamma^{(1)}:\Theta_f^{(1)}\to T\Theta_f^{(1)}\} \tag{17}$$

In terms of diagrams, for $\gamma \in \mathcal{G}_f$ we have

$$\begin{array}{ccc} \Theta_f^{(1)} & \xrightarrow{\gamma^{(1)}} & T\Theta_f^{(1)} \\ \downarrow \pi & & \downarrow d\pi \\ \Theta_f & \xrightarrow{\gamma} & T\Theta_f \end{array} \tag{18}$$

We stress that our definition (16), (17) *differs* from the usual one, which considers Γ_f instead of Θ_f. Again, $\mathcal{G}_f$ has the algebraic structure of a module, this time over the algebra I^f of function constants along the trajectory of f; clearly, if f is a solution to (1), then $I_0 \subseteq I^f$.

Clearly, a function $f: \mathbf{R}_+ \to M$ is a solution to (1) if and only if $\Theta_f^{(1)} \in S_0$.

4. LPTI SYMMETRIES AND TOPOLOGY OF SOLUTIONS

We stress that considering LPTI symmetries instead of general LP ones, and correspondingly the invariance of trajectory rather than the graph of a solution, allows us to introduce topological considerations in our symmetry analysis. Indeed, let us consider three solutions $x_0(t)$, $x_p(t)$, and $x_g(t)$, respectively a stationary one, a periodic one, and a generic one.

The graphs of the three, Γ_0, Γ_p, and Γ_g, are all topologically equivalent to an open line in $\mathbf{R}_+ \times M$, i.e., $\Gamma_0 \simeq \Gamma_p \simeq \Gamma_g \simeq \mathbf{R}_+$. If instead we consider their trajectories Θ_0, Θ_p, and Θ_g, these are *not* topologically equivalent: $\Theta_0 \simeq \{e\}$, $\Theta_p \simeq S^1$, and $\Theta_g \simeq \mathbf{R}_+$.

If now we consider the action of diffeomorphisms on these, it is clear that general LP transformations, i.e., diffeomorphisms of $(\mathbf{R}_+ \times M)$, or even time-dependent diffeomorphisms of M, can map Γ_0 in Γ_p or Γ_g; if instead we just consider LPTI transformations, i.e., diffeomorphisms of M, then it is clear that "two trajectories which are topologically different cannot be mapped into each other by a Lie group transformation" (Bluman and Kumei, 1989, p. 154).

Now, by definition, $\mathcal{G}_0$ transforms solutions to (1) into (generally, different) solutions to (1); the above discussion shows the advantage of considering instead $\mathcal{G}_0$: this can be set in the form of the following:

Lemma I. The LPTI symmetry algebra of an autonomous ODE transforms stationary solutions into stationary solutions, and periodic solutions into periodic solutions.

If we focus on periodic solutions, it is also easy to see that LPTI transformations cannot change the period of solutions: consider the suspension

γ_* of $\gamma: M \to TM$ in $\mathbf{R}_+ \times M$; its action on Γ_f for f periodic of period T satisfies $\gamma_*(t, f(t)) = \gamma_*(t+T, f(t+T))$, so that f is transformed into a (generally different) function of the same period. We have therefore that the above lemma can be rewritten as follows:

Lemma I'. The LPTI symmetry algebra of an autonomous ODE transforms stationary solutions into stationary solutions, and periodic solutions into periodic solutions of the same period.

It is well known that a dynamical system like (1) can also exhibit quasiperiodic solutions; in this case the solutions fill densely a (topological) torus $T^k \subset M$, with $k \geq 2$, $k \in \mathbf{Z}$. The number k is also called the number of modes present in the quasiperiodic solution, so that a k-mode quasiperiodic solution $f: \mathbf{R}_+ \to M$ is such that $f: \mathbf{R}_+ \to \mathscr{T} \subset M$ and fill densely $\mathscr{T}$, with $\mathscr{T}$ topologically a k-torus, $\mathscr{T} \simeq T^k$.

By repeating the above discussion, and using the smoothness of diffeomorphisms as well as invertibility of Lie group transformations, we also get the following result:

Lemma II. The LPTI symmetry algebra of an autonomous ODE transforms k-mode quasiperiodic solutions into k-mode quasiperiodic solutions.

We remark that LPTI symmetries have also shown to be useful in connection to periodic solutions of autonomous ODE, allowing for a classification of periodic solutions based on knot theory (Gaeta, 1991*a*; [see also Crawford and Omohundro (1984), unknown to me while writing Gaeta (1991*a*); I thank Prof. Crawford for pointing out his paper to me].

5. EXISTENCE OF STATIONARY AND PERIODIC SOLUTIONS

We want to consider the relation existing between zero sets of vector fields in $\mathscr{G}_0$ and invariant submanifolds of M under the flow (1). This will also allow us to state results on the existence of stationary or periodic solutions.

For any vector field $\gamma = \phi^i(x)\, \partial_x$ in $\mathscr{M}$ (and in particular in $\mathscr{G}_0$), we consider its kernel K_γ,

$$K_\gamma = \{x \in M / \phi(x) = 0\} \subseteq M \tag{19}$$

For a subalgebra $\mathscr{M}_* \subseteq \mathscr{M}$, $K(\mathscr{M}_*)$ will be the intersection of K_γ over all the $\gamma \in \mathscr{M}_*$. We have then immediately the following result:

Lemma III. For any subalgebra $\mathscr{G}_* \subseteq \mathscr{G}$, the submanifold $K(\mathscr{G}_*) \subseteq \mathscr{M}$ is invariant under the flow of (1).

Indeed, for $x \in K(\mathscr{G}_*)$, $\phi^i(x) = 0$ for any $\gamma \in \mathscr{G}_*$, so that (8) reduces to the vanishing of the Lie derivative of ϕ along the flow of (1), $L_F\phi = 0$, and hence the lemma.

This kind of simple but very useful reduction lemma has often been considered in the linear case (i.e., for linear group actions) in the context of bifurcation theory; see, e.g., Golubitsky and Stewart (1985), Golubitsky *et al.* (1988), and Gaeta (1990); it was also the basis of the discussion given in Cicogna and Gaeta (1989). A nonlinear version like the present one is used in Cicogna (1990) and (Gaeta, 1991*b*).

We stress once again that to be applied the above lemmas do not require the knowledge of the full symmetry algebra $\mathscr{G}_0$; in particular, the algebra $\mathscr{G}_*$ considered in Lemma III could have only one generator.

From Lemma III we have immediately the following result:

Corollary. If a vector field $\gamma \in \mathscr{G}_0$ has isolated zeros, these correspond to stationary solutions of (1); if K_γ contains an isolated set diffeomorphic to a circle, either this is the trajectory of a periodic solution or it contains stationary points for (1).

More generally, we can have similar results if we accept the following:

Assumption on F. There is a compact set $K_F \subseteq M$ invariant under the flow of (1).

This, essentially, allows us not only to restrict to $K(\mathscr{G}_*)$, but to actually restrict to $K(\mathscr{G}_*) \cap K_F$, which is a compact set due to the compactness of K_F and the smoothness of $K(\mathscr{G}_*)$; this allows us again to make a statement on the existence there of stationary or periodic solutions on the basis of topological arguments. Indeed, it is obvious that we have the following:

Lemma IV. For any subalgebra $\mathscr{G}_* \subseteq \mathscr{G}_0$, the submanifold

$$K^0(\mathscr{G}_*) \equiv K(\mathscr{G}_*) \cap K_F \subseteq M$$

is either empty or compact and invariant under the flow of (1).

From this there follow easy corollaries similar to the one considered above:

Corollary A. If $K^0(\mathscr{G}_*)$ contains isolated points, these correspond to stationary solutions of (1); if $K^0(\mathscr{G}_*)$ has one-dimensional components isomorphic to [0, 1], on each of these there is a stationary solution to (1).

The first part is analogous to one of the statements seen in the previous corollary; the second simply follows from invariance of the interval.

Corollary B. If $K^0(\mathscr{G}_*)$ has two-dimensional components, on each of these lie periodic (possibly stationary) solutions to (1).

This just follows from the observation that these components are compact invariant two-dimensional manifolds, and the standard Poincaré–Bendixson theorem (Hirsch and Smale, 1978).

Corollary C. If $K^0(\mathcal{G}_*)$ has components isomorphic to an even-dimensional sphere S^{2n}, then on each of these lie stationary solutions to (1).

This just follows from invariance of the components under F and the well-known theorem on the existence of zeros of vectors field on even-dimensional spheres (Milnor, 1965; Guillemin and Pollack, 1974; Arnold, 1983). Similarly, using theorems on fixed points on disks, one has the following result:

Corollary D. If $K^0(\mathcal{G}_*)$ has components isomorphic to a disk D^{2n+1}, then on each of these lie stationary solutions to (1).

It should be remarked that the last two corollaries were already obtained in the discussion of linear symmetries (Cicogna, 1984; Cicogna and Degiovanni, 1984): since the argument presented there is purely topological, it immediately applies to the general nonlinear case.

We also remark that the connection between periodic orbits and Lie-theoretic properties of the equation was already considered in Wulfman (1974); as already recalled, results similar to the ones given here were obtained for linear symmetries (Cicogna and Gaeta, 1989; Golubitsky and Stewart, 1985; Golubitsky *et al.*, 1988).

We also notice that in a bifurcation-theoretic setting (Golubitsky *et al.*, 1988; Gaeta, 1990; Sattinger, 1979; Chow and Hale, 1982; Ruelle, 1989; Crawford, 1991; Crawford and Knobloch, 1991), the assumption on F considered in this section is completely natural, and it is also completely natural to assume that K_F contains a fixed point x_0, losing stability in the bifurcation; one can therefore repeat our reasoning considering $K_F \backslash x_0$ instead of K_F (similarly, one could consider the case of a more complicated, e.g., periodic, solution ω_0, with $\omega_0 \in K_F$, losing stability in the bifurcation, and consider $K_F \backslash \omega_0$). We will not consider the bifurcation setting here; for a related discussion see Cicogna (1990) and Gaeta (1991*b*).

REFERENCES

Arnold, V. I. (1983). *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer, Berlin.

Arnold, V. I. (1990). *Equations Differentielles Ordinaires*, 2nd ed., Mir, Moscow.

Bluman, G. W., and Kumei, S. (1989). *Symmetries and Differential Equations*, Springer, Berlin.

Chow, S. N., and Hale, J. (1982). *Methods of Bifurcation Theory*, Springer, Berlin.

Cicogna, G. (1984). *Bollettino Unione Matematica Italiana*, **3**, A-131.

Cicogna, G. (1990). *Journal of Physics A*, **23**, L1339.

Cicogna, G., and Degiovanni, M. (1984). *Nuovo Cimento*, **82B**, 54.
Cicogna, G., and Gaeta, G. (1989). *Nonlinear Analysis*, **13**, 475.
Crawford, J. D. (1991). *Reviews of Modern Physics*, **63**, 991.
Crawford, J. D., and Knobloch, K. (1991). *Annual Review of Fluid Mechanics*, **23**, 341.
Crawford, J. D., and Omohundro, S. (1984). *Physica D*, **13**, 161.
Gaeta, G. (1990). *Physics Reports*, **189**, 1.
Gaeta, G. (1991*a*). The Jones polynomial and periodic orbits of nonlinear oscillators, Preprint, CPTh Ecole Polytechnique; to appear in *International Journal of Theoretical Physics*.
Gaeta, G. (1991*b*). Reduction and equivariant branching lemma: Dynamical systems, evolution PDEs, and gauge theories, Preprint, CPTh Ecole Polytechnique; to appear in Acta Appl. Math.
Gaeta, G. (1992). *Symmetries of Nonlinear Equations and Physics*, in preparation.
Golubitsky, M., and Stewart, I. (1985). *Archive for Rational Mechanics and Analysis*, **87**, 107.
Golubitsky, M., Stewart, I., and Schaeffer, D. G. (1988). *Singularity and Groups in Bifurcation Theory*, Vol. II, Springer, New York.
Guillemin, V., and Pollack, A. (1974). *Differential Topology*, Prentice-Hall, Englewood Cliffs, New Jersey.
Hirsch, M., and Smale, S. (1978). *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York.
Milnor, J. (1965). *Topology from the Differential Viewpoint*, University of Virginia Press.
Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*, Springer, Berlin.
Ovsjannikov, L. V. (1962). *Group Properties of Differential Equations*, Novosibirsk, Russia.
Ruelle, D. (1989). *Elements of Differentiable Dynamics and Bifurcation Theory*, Academic Press, London.
Sattinger, D. H. (1979). *Group Theoretic Methods in Bifurcation Theory*, Springer, Berlin.
Stephani, H. (1989). *Differential Equations. Their Solution Using Symmetries*, Cambridge University Press, Cambridge.
Wulfman, C. E. (1974). *Journal of Physics A*, **12**, L73.